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Euler Type Partial Differential Operators on Real Analytic Functions

Nilambari Panda
Department of Basic Science, Aryan Institute of Engineering & Technology, Bhubaneswar

Ranjana Pradhan,
Department of Basic Science, Raajdhani Engineering College, Bhubaneswar

Abstract

We describe all Euler partial diff erential operators which act on the space of real ana-lytic functions and we identify them among the Taylor multipliers on these spaces. Partial diff erential operators of the form

$$f \mapsto \sum a_\alpha D^\alpha f, \qquad D^\alpha := D_1^{\alpha_1} \cdots D_d^{\alpha_d}, \qquad D_j(f)(x) := q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j)f(x),$$

where $q_{j,1}, q_{j,0} : (a_j, b_j) \to \mathbb{C}$, are called generalized Euler diff erential operators whenever all D_j are conjugate to the classical Euler diff erential ϑ, $\vartheta(f)(t) = tf'(t)$. We find criteria when a linear diff erential operator with analytic coeffi cients on the space of real analytic functions is a generalized Euler diff erential operators. It turns out that this happens for a wide variety of linear operators with variable coeffi cients. Using our earlier results on solvability of classical Euler operators of finite order we then study the question of surjectivity or "big image" for generalized Euler partial diff erential operators with analytic coeffi cients, i.e., global solvability of the considered equations in spaces of real analytic functions.

I. INTRODUCTION

Unfortunately, there is no general theory of linear partial diff erential equations with variable coeffi cients as good as the theory for constant coeffi cient operators (comp. [13], [12]). Thus it makes sense to look for and analyze nice simple classes of variable coeffi cients linear partial diff erential operators and in the present paper we study such a class in order to get global solvability of the corresponding equations in analytic functions. This will be the first main aim of the paper.

Let us take $D_1, \ldots, D_d$ to be general linear diff erential operators of order one with analytic coeffi cients of the form

$$D_j(f)(x) = D_j(x_j, \partial_j)(f)(x) = q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j),$$

where $q_{j,1}, q_{j,0} : (a_j, b_j) \to \mathbb{C}$ are real analytic functions (i.e., they belong to $\mathcal{A}(a_j, b_j)$) possibly *vanishing* at some points. We will characterize those operators $D_j : \mathcal{A}(a_j, b_j) \to \mathcal{A}(a_j, b_j)$ which are conjugate to the Euler diff erential operator

$$\vartheta : \mathcal{A}(k_-, k_+) \to \mathcal{A}(k_-, k_+), \qquad \vartheta(f)(x) = xf'(x),$$

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for some $-\infty \le k_- < k_+ \le \infty$ (see Theorem 3.1). It turns out that this holds if and only if $q_{j,1}$ is real valued and either has no zeros or has exactly one zero $u \in (a_j, b_j)$ with $q^1_{j,1}(u) = 1$ and $q_{j,0}(u) = 0$. Therefore there are quite a lot of operators D_j conjugate to ϑ. Clearly, in this case also the linear partial differential operator with analytic coefficients of the form

$$f \mapsto \sum_\alpha D^\alpha f := \sum_\alpha a_\alpha D^{\alpha_1}_1 D^{\alpha_2}_2 \dots D^{\alpha_d}_d f, \qquad \alpha \in \mathbb{N}^d,$$

is conjugate to the classical Euler partial differential operator

$$f \mapsto \sum_\alpha a_\alpha \vartheta^\alpha f := \sum_\alpha a_\alpha \vartheta^{\alpha_1}_1 \dots \vartheta^{\alpha_d}_d f,$$

where a_α are constants and $\vartheta_j(f)(x) = x_j \frac{\partial f}{\partial x_j}(x)$. The former operators we call *generalized Euler differential operators*. Please note that we have obtained in [8] a deep theory on surjectivity (or "big image") of classical Euler linear partial differential operators. Via the conjugation relation this automatically produces a corresponding theory for generalized Euler partial differential operators, see Section 4. Therefore we obtain surjectivity results for a wide class of linear partial differential operators with variable coefficients or results describing closed image of these opera- tors, i.e., results on global solvability of these linear partial differential operators with analytic coefficients.

More precisely, in [8] we characterized when Euler partial differential operators of finite order have "big" images in spaces of real analytic functions and, in particular, we found many examples of surjective operators of that type. Using the conjugation relation mentioned above we are able to characterize surjective generalized Euler differential operators of finite order on the space of real analytic functions $A(\prod_{j=0}^d (a_j, b_j))$, $-\infty \le a_j < b_j \le +\infty$, for all $q_{j,1}$ having no zero (see Corollary 4.2, Corollary 4.3). Moreover, in case all $q_{j,1}$ have exactly one zero (and of order one) then we characterize when the corresponding generalized Euler partial differential operators of finite order on $A(\prod_{j=0}^d (a_j, b_j))$ have a "big" image (see Corollary 4.6).

Please note that surprisingly the conjugation in Theorem 3.1 can always be given by a weighted composition operator $V = C_{\kappa,w}$ (so by substitution and multiplication, see Theorem 3.2),

$$C_{\kappa,w}(f)(x) = w(x)f(\kappa(x)), \tag{1}$$

where κ is a real valued analytic diffeomorphism and w is a non-zero complex valued weight function,

$$\kappa(x_1,\dots,x_d) = (\kappa_1(x_1),\dots,\kappa_d(x_d)), \quad w(x_1,\dots,x_d) = w_1(x_1)\dots w_d(x_d)$$

and κ_j are analytic diffeomorphisms. If $q_{j,1}$ has a zero at u_j then $\kappa_j(u_j) = 0$.

In Theorem 3.6 we also characterize when a general first order linear differential operator with analytic coefficients,

$$D(f)(x) = q_1(x)f^1(x) + q_0(x)f(x)$$

is conjugate to ϑ on some *invariant subspace* of D— this is proved under the additional assumptions that q_1 is real valued.

The classical Euler differential operators are extensively studied on spaces of analytic functions for one variable (see for instance, [3], [15], [16], [18], [19], [20]) and recently in that case results on surjectivity were obtained by the authors (see [7]). For many variables see our paper [8].

Euler operators belong to the widely studied class of diff erential equations with polynomial coef-ficients (see [1], [21]). Unfortunately, the authors could not find the basic facts on Euler partial diff erential equations in the literature (in the several variables case) — we fill this gap in Section 2 and this is the second main aim of the paper. In particular, we characterize in Theorem 2.2 those sequences $(a_\alpha)_{\alpha\in\mathbb{N}^d}$ for which the operator $\sum a_\alpha\theta^\alpha$ is well defined on the space of real analytic functions $A(\Omega)$, $\Omega\subset\mathbb{R}^d$ open, generalizing to several variables the earlier results of the same authors for the one variable case [5, Section 4].

Every monomial is an eigenvector of every Euler diff erential operator so they are examples of so-called *Hadamard type multipliers* (or Taylor multipliers, comp. [8]) studied extensively in [5], [6], [7] (for the one variable case) and in [10], [8], [9] (for the several variables case). We identify which multipliers are Euler diff erential operators (see Corollary 2.5).

Now, by analogy to classical Taylor multipliers, we call an operator $M : A(\Omega)\to A(\Omega)$ to be a *κ, w-multiplier* if for any $\alpha\in\mathbb{N}^d$ the function $w\kappa^\alpha$, $w\kappa^\alpha(x) := w(x)\kappa_1^{\alpha_1}(x_1)\ldots\kappa_d^{\alpha_d}(x_d)$, is an eigenvector of M. Here $\kappa_j : \mathbb{R}\to\mathbb{R}$ is an arbitrary analytic map without critical points with $\kappa_j(u_j) = 0$, $w(x) = w_1(x_1)\ldots w_d(x_d)$, where w has no zeros. It turns out that every κ, w-multiplier corresponds to a "classical" Taylor multiplier and vice versa via (1) and then generalized Euler diff erential operators with

$$q_{j,1}(t) = \frac{\kappa_j(t)}{\kappa_j'(t)},\qquad q_{j,0}(t) = -\frac{\kappa_j(t)w_j'(t)}{\kappa_j'(t)w_j(t)}$$

correspond to "classical" Euler diff erential operators. So the results for Euler diff erential operators proved in Section 2 and the results on multipliers proved in [10], [8] and [9] and the other papers mentioned above transfer in an obvious way to results on generalized Euler diff erential operatorsand κ, w-multipliers, respectively. We omit the details.

Let us recall that a linear continuous isomorphism $V : X\to Y$ *conjugates* the operator $T : Y\to Y$ with the operator $S : X\to X$ if and only if the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{T} & Y \\ V\uparrow & & V\uparrow \\ X & \xrightarrow{S} & X. \end{array}$$

Please note that in that case T and S have the same eigenvalues and, if V is a topological iso-morphism, with isomorphic eigenspaces and the same spectrum. Of course, conjugate operators are either both surjective or both not surjective. If X and Y are spaces of real analytic functions (as mostly in our paper) then every continuous linear isomorphism $V : X\to Y$ is automatically a topological isomorphism since the open mapping theorem works for surjective maps. Unfor-tunately, if Y is only a closed subspace of the space of real analytic functions then this is not always the case since Y might not be ultrabornological and so the open mapping theorem might not work (see the information on the space $A(\Omega)$ below).

Let us denote by M_a, $a\in\mathbb{R}^d$, the dilation operator

$$M_a(f)(x) := f(ax),$$

where as usual $ax := (a_1x_1,\ldots,a_dx_d)$ for $a = (a_1,\ldots,a_d)$ and $x = (x_1,\ldots,x_d)\in\mathbb{R}^d$. Define

$$\mathbb{R}_* := \mathbb{R}\setminus\{0\},\quad |z| = |(z_1,\ldots,z_d)| := |z_1| + \cdots + |z_d|,$$
$$\eta_\alpha(z) := z^\alpha := z_1^{\alpha_1}\cdot\cdots\cdot z_d^{\alpha_d},\quad \text{for any } \alpha\in\mathbb{N}^d,$$
$$\mathbb{O} := (0,\ldots,0),\quad \mathbb{1} := (1,\ldots,1).$$

Let us recall that the space of real analytic functions $A(\Omega)$ on an open set $\Omega \subset \mathbb{R}^d$ is endowed with its natural topology $\operatorname{ind}_U H(U)$, i.e., the locally convex inductive limit topology, where $U \subset \mathbb{C}^d$ runs through all complex neighborhoods of Ω. The space $A(\Omega)$ is both ultrabornological and webbed so the closed graph theorem for operators on $A(\Omega)$ and the open mapping theorem for surjective operators work. Nevertheless, a closed subspace of $A(\Omega)$ need not be ultrabornological so the open mapping theorem for operators from $A(\Omega)$ onto a closed subspace of $A(\Omega)$ need not be true. For more information on this space see the survey [4]. For analytic functionals see [23].

Let $\Omega \subset \mathbb{R}^d$ be an open set. We denote by $M(\Omega)$ the set of all classical (Hadamard) multipliers on $A(\Omega)$, i.e. the class of all continuous linear operators

$$M : A(\Omega) \to A(\Omega) \text{ such that } M(\xi^\alpha)(x) = m_\alpha x^\alpha \text{ for any } \alpha \in \mathbb{N}^d.$$

The sequence $(m_\alpha)_\alpha$ of eigenvalues related to the monomials is called the *multiplier sequence*.

The *dilation set* is defined as follows:

$$V(\Omega) := \{x : x\Omega \subset \Omega\} = \bigcap_{y\in\Omega} \{x : xy \in \Omega\}.$$

Clearly, if Ω is convex then $V(\Omega)$ is convex as well, see [5] and [10] for more details on these sets. The following central result from [10] will play an important role for classical multipliers:

The Representation Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be an open set. The map*

$$B : A(V(\Omega))'_b \to M(\Omega) \subset L_b(A(\Omega)), \qquad B(T)(g)(y) := \langle g(y\cdot), T\rangle, T \in A(V(\Omega))', g \in A(\Omega),$$

is a bijective continuous linear map with a sequentially continuous inverse and the multiplier sequence of $B(T)$ is equal to the sequence of moments of the analytic functional T, i.e. to $(\langle x^\alpha, T\rangle)_{\alpha\in\mathbb{N}^d}$.

Moreover, if M is a multiplier on $A(\Omega)$ then for any $y \in \Omega \cap \mathbb{R}^d_*$ the linear continuous functional T defined as

$$T = \delta_1 \circ M_y \circ M \circ M_{1/y} : A((1/y)\Omega) \to \mathbb{C}, \tag{2}$$

where $M_y(g)(\xi) := g(y\xi)$ and δ_1 denotes the point evaluation at $\mathbb{1} := (1,\ldots, 1)$, extends to a continuous functional on $A(V(\Omega))$ (not depending on y) and $M = B(T)$.

For non-explained notions from Functional Analysis see [22].

2 Euler partial diff erential operators

In this section we consider Euler diff erential operators of infinite (and finite) order of many vari- ables. The one variable theory is classical — although the authors could not find its multivariableanalogue in the literature but the proofs of the results in this section for the many variable case are mostly completely analogous to the one-dimensional situation. We present them here just for the sake of convenience.

We will use multiindices $\alpha = (\alpha_1,\ldots, \alpha_d) \in \mathbb{N}^d$. Moreover,

$$|\alpha| = \alpha_1 + \cdots + \alpha_d \quad \text{and} \quad \alpha! = \alpha_1!\alpha_2!\ldots\alpha_d!.$$

Let $E(z) = \sum_{\alpha\in\mathbb{N}^d} a_\alpha z^\alpha$, $z \in \mathbb{C}^d$, be a formal power series of d variables. We say that the associated Euler diff erential operator

$$E(\vartheta) = \sum_{\alpha\in\mathbb{N}^d} a_\alpha \vartheta^\alpha, \qquad \text{where } \vartheta^\alpha := \vartheta_1^{\alpha_1}\vartheta_2^{\alpha_2}\ldots\vartheta_d^{\alpha_d}$$

acts on $A(\Omega)$, $\Omega \subset \mathbb{R}^d$ an open subset, if for any $f \in A(\Omega)$ the series

$$\sum_{\alpha\in\mathbb{N}^d} a_\alpha \vartheta^\alpha(f)$$

is pointwise convergent on Ω.

We call an entire function $f \in H(\mathbb{C}^d)$ to be of *exponential type zero* if for any $\varepsilon > 0$ there is a constant C such that for any $z \in \mathbb{C}^d$ holds

$$|f(z)| \le C \exp(\varepsilon|z|),$$

where $|z| = |z_1| + \cdots + |z_d|$. The class of entire functions of exponential type zero will be denoted by $\mathrm{Exp}(\{0\})$.

Lemma 2.1 *A formal power series* $\sum_{\alpha\in\mathbb{N}^d} a_\alpha z^\alpha$ *of d variables represents an entire function of exponential type zero if and only if*

$$\forall\varepsilon > 0 \quad \sup_\alpha |a_\alpha| \frac{\alpha!}{\varepsilon^{|\alpha|}} < \infty. \tag{3}$$

Proof: The proof is as in the one variable case.

If $|a_\alpha| \le \frac{M\varepsilon^{|\alpha|}}{\alpha!}$ then

$$\left|\sum_{\alpha\in\mathbb{N}^d} a_\alpha z^\alpha\right| \le \sum_{\alpha\in\mathbb{N}^d} \frac{M\varepsilon^{|\alpha|}}{\alpha!} |z_1|^{\alpha_1} \cdot\ldots\cdot |z_d|^{\alpha_d} \le M \sum_{\alpha\in\mathbb{N}^d} \frac{(\varepsilon|z_1|)^{\alpha_1} \cdots (\varepsilon|z_d|)^{\alpha_d}}{\alpha_1! \cdot\ldots\cdot \alpha_d!}$$
$$= M \exp(\varepsilon|z|).$$

Now, assume that f is of exponential type zero, then

$$|a_\alpha| \le \frac{1}{(2\pi)^d} \int_C \frac{|f(t)|}{|t_1|^{\alpha_1+1} \ldots |t_d|^{\alpha_d+1}} |dt_1| \ldots |dt_d|,$$

where C is the distinguished boundary of the polydisk at zero of polyradius $(R_1, \ldots, R_d)$. Choose $R_j = \frac{\alpha_j}{\varepsilon}$ then by Stirling's formula

$$|a_\alpha| \le C_1 \frac{\exp(|\alpha|)}{\prod_{j=1}^d \alpha_j^{\alpha_j}} \varepsilon^{|\alpha|} \le C_2 \frac{(2\varepsilon)^{|\alpha|}}{\alpha!}.$$

Q

Now, we present the multivariable analogue of [5, Th. 4.1].

Theorem 2.2 *Let* $E(\vartheta) = \sum_{\alpha\in\mathbb{N}^d}^{\infty} a_\alpha \vartheta^\alpha$ *be an Euler differential operator. The following assertions are equivalent.*

(a) *$E(\vartheta)$ acts on $A(\Omega)$ for every open non-empty set $\Omega \subseteq \mathbb{R}^d$.*

(b) *$E(\vartheta)$ acts on $A(\Omega)$ for some non-empty open set $\Omega \subseteq \mathbb{R}^d$.*

(c) *The series $\sum_{\alpha\in\mathbb{N}^d} a_\alpha z^\alpha$ is convergent on $\mathbb{C}^d$ and the corresponding entire function $E(z)$ is of exponential type zero.*

(d) *For any $x \in \mathbb{R}^d$ and any function $f \in A(\{x\})$ there is a neighbourhood $U \subset \mathbb{C}^d$ of x such that the series $\sum_{\alpha\in\mathbb{N}^d} a_\alpha \vartheta^\alpha f$ is uniformly convergent on U.*

(e) For every open non-empty set $\Omega \subset \mathbb{R}^d$ the map $E(\vartheta) : A(\Omega) \to A(\Omega)$ is a Hadamard multiplier with multiplier sequence $(E(\alpha))_{\alpha\in\mathbb{N}^d}$.

Remark 2.3 The implications (e)⇒(a)⇒(b) are obvious. The implication (b)⇒(c) is proved for one variable in [5, Th. 4.1]. The implication (c)⇒(d) and (e) is known for the one variable case, see [11, Th. 11.2.3]. The proof of (b)⇒(c)⇒(d) for the many variables case is similar to the one variable analogue — we present it for the sake of convenience.

First we need a lemma.

Lemma 2.4 *Let $a, z \in \mathbb{C}$, $a \neq z$, $\vartheta(f)(x) = xf'(x)$. Then*

$$\vartheta^n \frac{1}{a-z} = \frac{P_n(z,a)}{(a-z)^{n+1}},$$

where P_n is a two-variable polynomial such that

- $|P_n(z,a)| \le n!r^n$, *whenever* $\max(|z|,|a|) \le r$;
- $|P_n(1+i\varepsilon,1)| \ge n!$ *for any* $\varepsilon \in \mathbb{R}$.

Proof: The first part is proved in [11, Lemma 11.2.1, Corollary]. The last part is proved in the proof of [5, Th. 4.1, proof (b)⇒(c)]. Q

Proof of Theorem 2.2. (b)⇒(c): We define

$$g_\varepsilon(z) := \prod_{j=1}^{d} \frac{1}{1-\frac{(1+i\varepsilon)z_j}{w_j}}$$

for some fixed $w \in \Omega \cap \mathbb{R}^d$, $\varepsilon > 0$. Clearly, $g_\varepsilon \in A(\mathbb{R}^d) \subset A(\Omega)$. If

$$f(z) := \prod_{j=1}^{d} \frac{1}{1-z_j}$$

then since the dilation M_b, $b = \frac{(1+i\varepsilon)}{w}$, and the differential operator ϑ_j commute we get

$$\vartheta^\alpha(g_\varepsilon)(w) = \vartheta^\alpha(f)(1+i\varepsilon, \ldots, 1+i\varepsilon).$$

Hence, by Lemma 2.4,

$$|\vartheta^\alpha(g_\varepsilon)(w)| = \left|\prod_{j=1}^{d} \frac{P_{\alpha_j}(1+i\varepsilon,1)}{(-i\varepsilon)^{\alpha_j+1}}\right| \ge \frac{\alpha!}{\varepsilon^{|\alpha|+d}}.$$

Therefore, if $\sum_\alpha a_\alpha \vartheta^\alpha(g_\varepsilon)(w)$ converges then

$$\sup_{\alpha\in\mathbb{N}^d} |a_\alpha| \frac{\alpha!}{\varepsilon^{|\alpha|}} < \infty.$$

By Lemma 2.1, E is an entire function of exponential type zero.

(c)⇒(d): Let f be a holomorphic function on a neighbourhood of $x \in \mathbb{R}^d$ containing the polydisk D centered at x with polyradius $(2\delta, \ldots, 2\delta)$ and distinguished boundary C. Let z be contained in the polydisk D_1 centered at x with polyradius $(\delta, \ldots, \delta)$. Then

$$f(z) = \frac{1}{(2\pi i)^d} \int_C \frac{f(t)}{(t_1-z_1)\ldots(t_d-z_d)} dt_1 \ldots dt_d.$$

By Lemma 2.4,

$$\sum_{m\le|\alpha|\le n} a_\alpha \vartheta^\alpha f(z) = \frac{1}{(2\pi i)^d}\int_C \sum_{m\le|\alpha|\le n} a_\alpha \prod_{j=1}^{d} \frac{P_{\alpha_j}(z_j,t_j)}{(t_j-z_j)^{\alpha_j}} \frac{f(t)}{\prod_{j=1}^{d}(t_j-z_j)} dt_1\cdots dt_d.$$

Choose r so big that $\max(|z_j|,|t_j|)\le r$ for all j and z,t in the formula above. By the assumption, there is a constant M such that

$$|a_\alpha| \le \frac{M}{\alpha!}\left(\frac{\delta}{2r}\right)^{|\alpha|}.$$

Hence, by Lemma 2.4, for $z\in D_1$ we have

(4) $$\left|\sum_{m\le|\alpha|\le n} a_\alpha \vartheta^\alpha f(z)\right| \le \frac{1}{(2\pi)^d}\int_C \sum_{m\le|\alpha|\le n} \frac{M}{2^{|\alpha|}} \frac{|f(t)|}{\prod_{j=1}^{d}|t_j-z_j|}|dt_1|\cdots|dt_d|$$
$$\le M_1 \sup_{z\in D}|f(z)|\frac{2^d m^d}{2^m} \xrightarrow[m\to\infty]{} 0,$$

where the constant M_1 does not depend on f.

(d)⇒(e): The assumption means that $E(\vartheta)$ maps holomorphic germs around x to holomorphic germs around x. By the closed graph theorem [17, Th. 3.3.4] applied to $A(\{x\})$ this map is continuous. By Grothendieck's factorization theorem [22, 24.33], for any neighbourhood $V\subset \mathbb{C}^d$ of x there is a neighborhood $U\subset\mathbb{C}^d$ of x such that $E(\vartheta)$ maps $H(V)$ to $H(U)$ continuously. Hence, $E(\vartheta)$ maps $A(\Omega)$ to $A(\Omega)$. By the closed graph theorem applied to $A(\Omega)$ (the space $A(\Omega)$ is webbed ultrabornological, see [4, Lecture 1]), $E(\vartheta): A(\Omega)\to A(\Omega)$ is continuous. Clearly,

$$E(\vartheta)(\eta_\alpha) = E(\alpha)\eta_\alpha.$$

Q

The next result is known for the one variable case (see [5, Cor. 4.3]) and the proof can be easily transferred to the multivariable case except for the implication (b)⇒(c) where we have to substitute Wigert's theorem available only for one variable. Essentially the result means that Euler differential operators are exactly the multipliers with support in $\{1\}$ — that means the only ones which act on all open non-empty sets. We present all the proofs for the sake of convenience.

Corollary 2.5 *Let M be a linear map on the space of polynomials of d variables. The following assertions are equivalent.*

(a) M extends to a multiplier on $A(\Omega)$ for every open non-empty set $\Omega\subset\mathbb{R}^d$.

(b) There is an analytic functional $T\in A(\{1\})'$ such that for every polynomial p holds:

$$M(p)(y) = \langle p(y\,\cdot\,), T\rangle.$$

(c) There is an entire function $E\in H(\mathbb{C}^d)$ of exponential type zero such that $M=E(\vartheta)$, i.e., M is an Euler differential operator with multiplier sequence $(E(\alpha))_{\alpha\in\mathbb{N}^d}$.

Proof: (a)⇒(b): We have that $M: A(\Omega)\to A(\Omega)$ for every convex open non-empty set $\Omega\subset\mathbb{R}^d$, in particular, for some Ω with $V(\Omega)=\{1\}$. Thus the Representation Theorem 1.1 implies (b).

(b)⇒(c): We define

$$E(z) := \langle \eta_z, T\rangle,$$

where $\eta_z(x)=\exp(z_1\log x_1+\dots+z_d\log x_d)$ for x close to 1. Clearly, $(E(\alpha))_{\alpha\in\mathbb{N}^d}$ is the multiplier sequence for M.

By Theorem 2.2, it suffices to show that E is entire of exponential type zero since then $E(\vartheta): A(\Omega)\to A(\Omega)$ is a linear continuous multiplier extending M (with the same multiplier sequence).

Since T is an analytic functional with support in $\{1\}$, by a several variable version of the Köthe-Grothendieck duality [24] the functional T corresponds a holomorphic function $f\in H_0((\hat{\mathbb{C}}\setminus\{1\})^d)$, such that

$$\langle g, T\rangle = \frac{1}{(2\pi i)^d}\int_{\gamma_1}\cdots\int_{\gamma_d} f(w)g(w)dw,$$

where γ_j is a curve in $\hat{\mathbb{C}} \setminus \{1\}$ of index 1 with respect to the point 1 and $\gamma_1 \times \cdots \times \gamma_d$ is contained in the domain of definition of both f and g. Since η_z is defined on $\{w : \forall j \ \operatorname{Re} w_j > 0\}$ so the choice of $\gamma_1 \times \cdots \times \gamma_d$ does not depend on z for calculating $\langle \eta_z, T \rangle$. Thus E is entire. Finally, if $|f(z)| \leq C$ on $\gamma_1 \times \cdots \times \gamma_d$ we have

$$|E(z)| \leq \frac{C}{(2\pi)^d} \int_{\gamma_1} \cdots \int_{\gamma_d} \exp\left(\sum_{j=1}^{d} (\operatorname{Re} z_j \log|w_j| - \operatorname{Im} z_j \arg w_j)\right) |dw|.$$

We can choose $\gamma_1, \ldots, \gamma_d$ so small that $|\arg w_j| \leq \varepsilon$ and $\log|w_j| \leq \varepsilon$, hence

$$|E(z)| \leq C \exp(2\varepsilon|z|).$$

(c)⇒(a): Follows from Theorem 2.2. Q

3 Conjugation relation

Now, we consider a general linear ordinary differential operator with variable coefficients D, $D(f)(x) := q_1(x)f'(x) + q_0(x)f(x)$, where $q_1, q_0 : (a, b) \to \mathbb{C}$ are arbitrary real analytic functions. We clarify the relation of this operator to the classical Euler operator ϑ, $\vartheta(f)(x) = xf'(x)$ on spaces of real analytic functions. Please recall that a linear continuous isomorphism $V : X \to Y$ conjugates the operator $T : Y \to Y$ with the operator $S : X \to X$ if and only if the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{T} & Y \\ V\uparrow & & V\uparrow \\ X & \xrightarrow{S} & X. \end{array}$$

Please note that in that case T and S have the same eigenvalues, and in case V is a topological isomorphism, with isomorphic eigenspaces and the same spectrum. The main theorem of this paper is the following one:

Theorem 3.1 *A general linear differential operator of first order $D : A(a, b) \to A(a, b)$, $D(f)(x) := q_1(x)f'(x)+q_0(x)f(x)$, where $q_1, q_0 : (a, b) \to \mathbb{C}$, $-\infty \leq a < b \leq +\infty$, be real analytic functions, q_1 non-constantly zero, is conjugate to the Euler differential $\vartheta : A(c, d) \to A(c, d)$ for some $-\infty \leq c < d \leq +\infty$ if and only if the map q_1 is real valued and if $0 \in (c, d)$ then q_1 has exactly one zero $u \in (a, b)$, $q_1^1(u) = 1$, $q_0(u) = 0$ while if $0 \notin (c, d)$ then q_1 has no zero.*

In fact, we can prove much more determining the conjugation map in Theorem 3.1:

Theorem 3.2 *Let $q_1, q_0 : (a, b) \to \mathbb{C}$, $-\infty \leq a < b \leq +\infty$, be real analytic functions, q_1 non-constantly zero. Define a general linear ordinary differential operator of first order D:*

$$D(f)(x) := q_1(x)f'(x) + q_0(x)f(x).$$

The following assertions are equivalent.

(a) *The general linear differential operator of first order $D : A(a, b) \to A(a, b)$ is conjugate to $\vartheta : A(c, d) \to A(c, d)$ for some $-\infty \leq c < d \leq +\infty$.*

(b) *The map q_1 is real valued. If $0 \in (c, d)$ then q_1 has exactly one zero $u \in (a, b)$ and $q_1^1(u) = 1$, $q_0(u) = 0$. If $0 \notin (c, d)$ then q_1 has no zero.*

(c) *There is an analytic diffeomorphism*

$$\kappa : (a, b) \to (k_-, k_+), \qquad -\infty \leq k_- < k_+ \leq +\infty,$$

and a never vanishing weight $w : (a, b) \to \mathbb{C}$ such that the weighted composition operator

$$C_{\kappa,w} : A(k_-, k_+) \to A(a, b), \qquad C_{\kappa,w}(f)(x) = w(x)f(\kappa(x)),$$

conjugates $D : A(a, b) \to A(a, b)$ with $\vartheta : A(k_-, k_+) \to A(k_-, k_+)$.

Moreover, if these conditions hold then $n \in \mathbb{N}$ is an eigenvalue with the eigenspace spanned by $\kappa^n w$ and if q_1 has a zero then these are the only eigenvalues.

If q_1 has no zero then $0 \le k_- < k_+ \le +\infty$ and for any $v \in (a, b)$:

- *if $q_1 > 0$ then $k_- = 0$ if and only if $\int_a^v \frac{1}{q_1(t)} dt$ is divergent and $k_+ = +\infty$ if and only if $\int_v^b \frac{1}{q_1(t)} dt$ is divergent;*
- *if $q_1 < 0$ then $k_- = 0$ if and only if $\int_v^b \frac{1}{q_1(t)} dt$ is divergent and $k_+ = +\infty$ if and only if $\int_a^v \frac{1}{q_1(t)} dt$ is divergent.*

If $q_1(u) = 0$ for some $u \in (a, b)$ then $-\infty \le k_- < 0 < k_+ \le +\infty$ and

- *$k_+ = +\infty$ if and only if $\int_v^b \frac{1}{q_1(t)} dt$ is divergent for $v \in (u, b)$;*
- *$k_- = -\infty$ if and only if $\int_a^v \frac{1}{q_1(t)} dt$ is divergent for $v \in (a, u)$.*

Proof: (a)⇒(b): Let us denote by V the conjugation map, i.e., a continuous isomorphism $V : A(c, d) \to A(a, b)$ such that

$$\begin{array}{ccc} A(a,b) & \xrightarrow{\xi} & A(a,b) \\ V\uparrow & & \uparrow V \\ A(c,d) & \xrightarrow{\vartheta} & A(c,d) \end{array}$$

is commutative. Clearly, since all natural numbers are eigenvalues of ϑ, so they are also eigenvalues for D. Moreover, the corresponding eigenspaces are one-dimensional.

We denote by η_n, $\eta_n(x) = x^n$ and set $w := V(\eta_0)$. We choose an open interval $I \subset (a, b)$ such that $q_1(x) = 0$ for any $x \in I$ and set for fixed $y \in I$ such that $w(y) \neq 0$

$$\kappa(x) := \exp \int_y^x \frac{1}{q_1(t)} dt \qquad \text{for } x \in I.$$

κ is a real analytic function on I and $\kappa(y) = 1$. Set $\kappa_n := \kappa^n w$ on I.

(i): There are $a_n \in \mathbb{C}$ such that $V(\eta_n) = a_n \kappa_n$ on I for any $n \in \mathbb{N}$.

Proof. We first prove that κ_n is an eigenvector of D on I with the eigenvalue $n \in \mathbb{N}$. Notice that

(5) $$q_1\kappa' = \kappa \text{ on } I \text{ and } q_1 w' = -q_0 w \text{ on } (a, b).$$

Hence we get on I

$$q_1\kappa_n' = nq_1\kappa^{n-1}\kappa' w + q_1\kappa^n w' = n\kappa^n w - q_0\kappa^n w = (n - q_0)\kappa_n.$$

Since also $DV(\eta_n)|_I = nV(\eta_n)|_I$, and the eigenspace of D for n in $A(I)$ is one-dimensional, the claim follows.

(ii): The function q_1 is real valued on (a, b).

Proof. We first prove that κ is real valued on I. Since $\kappa'(y) = 1/q_1(y) \neq 0$ we may shrink I (keeping $y \in I$) such that $\kappa : I \to \mathbb{C}$ is an analytic diffeomorphism onto a curve $\gamma \subset \mathbb{C}$ (use the holomorphic inverse function theorem) and w does not vanish on I. Thus the weighted composition operator

$$C_{\kappa,w} : A(\gamma) \to A(I), \qquad C_{\kappa,w}(f)(x) := w(x)f(\kappa(x)), \quad x \in I,$$

is a topological isomorphism with inverse

$$S := C_{\kappa^{-1}, \frac{1}{w\circ\kappa^{-1}}} : A(I) \to A(\gamma), \qquad S(f)(x) = \frac{1}{w(\kappa^{-1}(x))} f(\kappa^{-1}(x)), \quad x \in \gamma.$$

With $R(g) := g|_I$ for $g \in A(a, b)$ we set

$$L := S \circ R \circ V : A(c, d) \to A(\gamma).$$

Then L is continuous and linear and we get by (i)

$$L(P)(x) = S \circ R \circ V(P)(x) = C_{\kappa^{-1}, \frac{1}{w\circ\kappa^{-1}}}\left(\left(\sum_{j=0}^m a_j b_j \kappa^j w\right)\Big|_I\right)(x) = \sum_{j=0}^m a_j b_j x^j, \; x \in \gamma,$$

for $P(x) = \sum_{j=0}^m b_j x^j$. Since $1 = \kappa(y) \in \gamma$ we can apply the evaluation δ_1 at 1 and get

$$T := \delta_1 \circ L : A(c, d) \to \mathbb{C}, \qquad \langle T, \eta_n \rangle = a_n \quad \text{for every } n \in \mathbb{N}.$$

Notice that T is a continuous analytic functional with supp $T \subset (c, d)$.

If there is $y_1 \in I$ such that $\zeta = \kappa(y_1) \notin \mathbb{R}$ then for

$$M_\zeta(f)(z) := f(\zeta z), \qquad M_{1/\zeta}(f)(z) := f\left(\frac{z}{\zeta}\right),$$

we have the following continuous linear functional

$$T_1 := \delta_1 \circ M_\zeta \circ L \circ M_{1/\zeta} : A((1/\zeta)(c, d)) \to \mathbb{C}, \qquad \langle T_1, \eta_n \rangle = a_n \text{ for every } n \in \mathbb{N}.$$

Please note that

$$M_\zeta \circ L \circ M_{1/\zeta} : A((1/\zeta)(c, d)) \to A((1/\zeta)\gamma)$$

and $\frac{\kappa(y1)}{\zeta} = 1 \in (1/\zeta)\gamma$ so T_1 is a well defined analytic functional with supp $T_1 \subset (1/\zeta)(c, d)$.

Since T and T_1 have the same moments they are equal on entire functions which are dense both in $A(c, d)$ and in $A((1/\zeta)(c, d))$. Therefore $T = T_1$ on the space of entire functions and T has two convex compact carriers

$$K \subset (c, d) \quad \text{and} \quad K_1 \subset (1/\zeta)(c, d).$$

By the proof of [14, Theorem 4.5.3] for $n = 1$ the Borel transform B of the Fourier transform $\hat{T}$ can be analytically extended to $\mathbb{C}\setminus K$ and to $\mathbb{C}\setminus K_1$, hence to $\mathbb{C}\setminus(K \cap K_1) \supset \mathbb{C}\setminus((c, d)\cap(c, d)/\zeta)$. If $0 \notin (c, d)$ then $(c, d) \cap (c, d)/\zeta = \emptyset$, hence B is an entire function, that is, $\text{supp}(T) = \emptyset$; a contradiction. If $0 \in (c, d)$ then $(c, d) \cap (c, d)/\zeta = \{0\}$ and $\text{supp}(T) = \{0\}$.

Since $\langle T, \eta_n \rangle = a_n$ we then have

$$\forall\, R > 0 \quad \sup_n R^n |a_n| < \infty.$$

Since $L(\eta_n) = a_n\eta_n$ this implies that L acts continuously from $H(\{0\})$ into $H(\mathbb{C})$. Therefore

$$L(A(c, d)) \subset L(H(\{0\})) \subset H(\mathbb{C}).$$

This is impossible. Indeed, for $f \in A(a, b)$ choose $g \in A(c, d)$ such that $V(g) = f$. Then

$$L(g)(\kappa(x))w(x) = (C_{\kappa,w} \circ L)(g)(x) = V(g)(x) = f(x), \qquad x \in I.$$

Since $L(g)$ is entire, the left hand side is holomorphic on a fixed complex neighborhood of I (independent of f), a contradiction.

Hence κ and κ' are real valued on I. By (5), $q_1(x) = \kappa(x)/\kappa'(x) \in \mathbb{R}$, for $x \in I$ (notice that $\kappa'(x) \neq 0$ for $x \in I$ by (5)). Hence q_1 is real valued on (a, b) since q_1 is real analytic.

(iii): Let $q_1(u) = 0$ for some $u \in (a, b)$. Then $q_1'(u) = 0$ and $q_0(u) = 0$.

Proof. The functions $f_n := V(\eta_n)$ are eigenvectors of D for $n \in \mathbb{N}$, that is,

$$q_1(x)f_n'(x) = (n - q_0(x))f_n(x), \qquad x \in (a, b). \tag{6}$$

Choosing $n = q_0(u)$ in (6) we see that $q_1'(u) = 0$ because the order of zero at u of f_n' plus one is exactly equal to the order of zero at u of f_n. Also by (6)

$$f_n(x) = f_n(c)\exp\left(\int_c^x \frac{n - q_0(t)}{q_1(t)}\,dt\right), \quad u < x \le c.$$

if $q_1(x) \neq 0$ for $u < x \le c$. Please note that

$$f_n'(x) = f_n(c)\frac{n - q_0(x)}{q_1(x)}\exp\left(\int_c^x \frac{n - q_0(t)}{q_1(t)}\,dt\right), \quad u < x < c.$$

The function f_n^1 is real analytic on (a, b), hence the function on the right hand side can be extended analytically across u. Since q_1 is real valued by (ii) we get

$$(7) \qquad 0 = \lim_{x\to u^+} \exp \int_c^x \frac{n - q_0(t)}{q_1(t)}\, dt = \lim_{x\to u^+} \exp \int_c^x \frac{n - \operatorname{Re} q_0(t)}{q_1(t)}\, dt \quad \text{if } q_0(u) \neq n$$

since $q_1^1(u) = 0$ and hence $(n - q_0)/q_1$ then has a pole in u.

If $q_0(u) \notin \mathbb{N}$ we get $0 = f_n(u) = V(\eta_n)(u)$ for any $n \in \mathbb{N}$, hence $V(P)(u) = 0$ for any polynomial P. Since the polynomials are dense in $A(c, d)$ we get $V(f)(u) = 0$ for any $f \in A(c, d)$ and V is not surjective; a contradiction.

If $q_0(u) = j$ for some $1 \leq j \in \mathbb{N}$ then (7) implies that

$$(8) \qquad -\infty = \lim_{x\to u^+} \int_c^x \frac{n - \operatorname{Re} q_0(t)}{q_1(t)}\, dt = \lim_{x\to u^+} (j - n) \int_x^c \frac{1}{q_1(t)}\, dt \text{ if } n \neq j$$

since $(\operatorname{Re} q_0 - j)/q_1$ is continuous near u because $q_1^1(u)$ 0. The formula (8) implies that $(j - n)/q_1^1(u) < 0$ for $n = 0$ and $n = j + 1$, that is, $j/q_1^1(u) < 0$ and $-1/q_1^1(u) < 0$; a contradiction.

(iv) : The function $w := V(\eta_0)$ has no zeros.

Proof. Please note that $w := V(\eta_0)$ satisfies (5). Also, $w(x) \neq 0$ for any $x \in (a, b)$. Indeed, if $w(v) = 0$ then $q_1(v) = 0$ by (5), hence $q_0(v) = 0$ by (iii) and the order of v for q_1 is at least 2 by (5); a contradiction to (iii).

(v) : The function q_1 has a zero at some $u \in (a, b)$ if and only if $0 \in (c, d)$.

Proof. Observe that w spans $\ker D$.

We determine $\dim \ker(D^2)$. If $f \in \ker D^2$ then $Df \in \ker(D)$, hence there is $r \in \mathbb{C}$ such that

$$(Df)(x) = q_1(x)f^1(x) + q_0(x)f(x) = rw(x).$$

Clearly, $f(x) = C(x)w(x)$ for some real analytic function C since w has no zeros (see (iv)). Thus

$$rw(x) = q_1(x)(Cw)^1(x) + q_0(x)C(x)w(x) = q_1(x)C^1(x)w(x)$$

by (5) and hence

$$q_1(x)C^1(x) = r \text{ for any } x \in (a, b).$$

If q_1 has a zero at $u \in (a, b)$ then this implies $r = 0$, and then $C^1 \equiv 0$. Thus $f \in \operatorname{span}(w)$ and $\dim \ker(D^2) = 1$.

If q_1 has no zero on (a, b) then $C^1(x) = r/q_1(x)$, thus

$$f(x) = \left(\int^x \frac{r}{q_1(t)}\, dt + C_1 \right) w(x), \qquad \text{for some } r, C_1 \in \mathbb{C},$$

and $\dim \ker(D^2) = 2$.

Of course, the same applies to ϑ (for $q_1(x) := x$ and $q_0 \equiv 0$). Thus if $0 \in (c, d)$ then $\dim \ker(\vartheta^2) = 1$ and if $0 \notin (c, d)$ then $\dim \ker \vartheta^2 = 2$.

By conjugation, $\dim \ker(\vartheta^2) = \dim \ker(D^2)$. This completes the proof.

(vi): If $q_1(u) = 0$ for some $u \in (a, b)$ then $f_1(u) = 0$ and $q_1^1(u) = 1/m$ where m is the order of u for f_1. Hence q_1 has at most one zero on (a, b).

Proof. Recall that $f_1 := V(\eta_1)$. Let $q_1(u) = 0$. Then $q_1^1(u) = 0$ by (iii), i.e. $q_1(x) = a_1(x - u) + a_2(x - u)^2 + \ldots$ with $a_1 \neq 0$. Also $f_1(u) = 0$ by (6) (for $n = 1$) since $q_0(u) = 0$ by (iii), i.e. $f_1(x) = b_m(x - u)^m + b_{m+1}(x - u)^{m+1} \ldots$ for some $1 \leq m \in \mathbb{N}$ and $b_m = 0$. Comparing the coeffi cients of $(x - u)^m$ on both sides of (6) we get $q_1^1(u) = a_1 = 1/m$. Thus $q_1^1(u) > 0$ at each zero u of q_1. Since q_1 is real valued by (ii), q_1 cannot have two diff erent zeros.

(vii) : If $q_1(u) = 0$ for some $u \in (a, b)$ then $q_1^1(u) = 1$.

Proof. We will show that $m = 1$ in (vi). Let $m > 1$, then $f_1(x) = (x - u)^m \phi(x)$ for some real analytic function ϕ with $\phi(u)$ 0. We may choose an interval I, $u \in I$, such that $f_1^{1/m}(x) := (x-u) \sqrt[m]{\phi(x)}$ and $w^{1-1/m}(x)$ are well defined real analytic functions on I (recall that $w := V(\eta_0)$ has no zeros on (a, b) by (iv)). Set

$$g(x) := f_1^{1/m}(x) w^{1-1/m}(x) \text{ for } x \in I.$$

Using (6) (for $n = 1$) we get for $u \neq x \in I$

(9) $$q_1(x)(f_1^{1/m})'(x) = q_1(x)\frac{1}{m} f_1^{1/m-1}(x) f_1'(x) = \frac{1}{m} f_1^{1/m}(x)(1 - q_0(x))$$

and hence $q_1(f_1^{1/m})' = \frac{1}{m} f_1^{1/m}(1 - q_0)$ on I since the left and the right hand side of (9) are continuous on I. Similarly, (5) implies that

(10) $$q_1(x)(w^{1-1/m})'(x) = -\left(1 - \frac{1}{m}\right) w^{1-1/m}(x) q_0(x).$$

Hence

$$q_1 g' = \frac{1}{m} f_1^{1/m}(1 - q_0) w^{1-1/m} - f_1^{1/m}\left(1 - \frac{1}{m}\right) w^{1-1/m} q_0 = \left(\frac{1}{m} - q_0\right) g \text{ on } I.$$

Since u is the only zero of q_1 the usual solution formula for the initial value problem

$$q_1 y' = \left(\frac{1}{m} - q_0\right) y, \quad y(c) = g(c)$$

for $u \neq c \in I$ provides an analytic extension $\tilde{g}$ of g to (a, b). Hence $\frac{1}{m} \notin \mathbb{N}$ is an eigenvalue of D on $A(a, b)$ and then also of ϑ on $A(c, d)$. But $0 \in (c, d)$ by (v), so ϑ has only eigenvalues $n \in \mathbb{N}$; a contradiction.

(b) $\Rightarrow$(c): First consider the case when q_1 has no zero. Just take for a fixed $v \in (a, b)$

$$\kappa(x) := \exp\left(\int_v^x \frac{1}{q_1(t)} dt\right), \qquad w(x) := \exp\left(\int_v^x \frac{-q_0(t)}{q_1(t)} dt\right).$$

If $q_1 > 0$ (and if $q_1 < 0$) we set

$$k_- := \lim_{x\to a} \kappa(x), \quad k_+ := \lim_{x\to b} \kappa(x) \text{ (and } k_- := \lim_{x\to b} \kappa(x), \quad k_+ := \lim_{x\to a} \kappa(x), \text{ respectively).}$$

The rest is an obvious calculation. Please note that by the theory of ordinary diff erential equa- tions all solutions of the first order diff erential equation $Df = \lambda f$ are proportional.

Now, we consider the case when q_1 vanishes exactly at u (and $q_1'(u) = 1$, $q_0(u) = 0$). Then there exists a real analytic real valued nowhere vanishing function $r : (a, b) \to \mathbb{R}$ such that

$$q_1(x) = (x - u) r(x) \qquad \text{and } r(u) = 1,$$

since $r(u) = q_1'(u) = 1$. Hence $r(x) > 0$ (since r has no zeroes on (a, b)) for every $x \in (a, b)$. We define two real analytic functions κ and w on (a, b) by:

$$\kappa(x) := (x - u) \exp\left(\int_u^x \frac{1 - r(t)}{q_1(t)} dt\right), \qquad w(x) := \exp\left(\int_u^x \frac{-q_0(t)}{q_1(t)} dt\right)$$

Let us calculate the derivative for $x \in (a, b)$

(11) $$\kappa'(x) = \frac{1}{r(x)} \exp\left(\int_u^x \frac{1 - r(t)}{q_1(t)} dt\right) > 0.$$

Hence κ is a strictly increasing real valued analytic diff eomorphism and we define

$$k_- := \lim_{x\to a^+} \kappa(x), \qquad k_+ := \lim_{x\to b^-} \kappa(x).$$

Finally, we have $q_1\kappa' = \kappa$ by (11) and clearly $q_1 w' + q_0 w = 0$. Hence

$$\begin{aligned} D\, C_{\kappa,w}(f) &= q_1[w(f \circ \kappa)]' + q_0 w(f \circ \kappa) \\ &= (q_1 w' + q_0 w)(f \circ \kappa) + w(q_1\kappa')(f' \circ \kappa) \\ &= C_{\kappa,w}\, \vartheta(f) \;. \end{aligned}$$

Since $q_1\kappa' = \kappa$ we have for any fixed c with $u < c < b$

$$\kappa(x) = \kappa(c) \exp\left(\int_c^x \frac{1}{q_1(t)} dt\right), \quad x \in (c, b),$$

hence $k_+ = +\infty$ if and only if $\int_c^x \frac{1}{q_1(t)} dt$ is divergent. Please note that since $q_1'(u) = 1$, for $x > u$ we have $q_1(x) > 0$ and for $x < u$ we have $q_1(x) < 0$. Therefore divergence of the considered integral always means divergence to $+\infty$.

The proof for k_- is analogous. Finally, it is well-known that the eigenvalues for ϑ in $A(k_-, k_+)$ are natural numbers whenever $k_- < 0 < k_+$. So the same holds for the conjugate operator D and the eigenspaces are spanned by $C_{\kappa,w}(\eta_n) = \kappa^n w$.

(c) ⇒(a): Obvious. Q

The authors believe that if D is of order higher than one then D cannot be conjugate to ϑ but we cannot prove it except in the somehow "trivial" case when the coefficient at the highest order term has no zeros.

Remark 3.3 If $q_1 : (a, b) \to \mathbb{R}$, $u \in (a, b)$, $q_1(u) = 0$, $q_1^1(u) = 1$, $q_0(u) = 0$ has an additional (second) zero at some $v \in (a, b)$ then there is no analytic eigenvector of D defined at v. Indeed, assume that $v \in (u, b)$ then, by Theorem 3.2 applied to the interval (a, v), κ tends to $+\infty$ at v and hence it cannot be extended to the whole interval (a, b). Moreover, if q_1 has a zero of order at least 2 at u then there is no analytic eigenvector of D defined near u, see Proposition 3.4 below.

Under the general assumption that q_1 is real valued we can prove much more. In particular, we can consider conjugation of the restriction of D to some invariant subspace.

Proposition 3.4 *If $q_1 : (a, b) \to \mathbb{R}$ and $q_0 : (a, b) \to \mathbb{C}$ are real analytic, $-\infty \le a < b \le +\infty$, $u \in (a, b)$, $q_1(u) = q_1^1(u) = \cdots = q_1^{(n-1)}(u) = 0$, $q_1^{(n)}(u) = 0$, $n > 1$, then there is no real analytic non-constant eigenvector of D, $D(f)(x) = q_1(x)f^1(x) + q_0(x)f(x)$, in $A(a, b)$ defined on a neighborhood of u for all values $\lambda \in \mathbb{C}$ except possibly $\lambda = q_0(u)$.*

Proof: By the theory of ordinary differential equations such an eigenvector f for the first order differential operator D and for the eigenvalue $\lambda \in \mathbb{C}$ must be of the form

$$f(x) = C \exp \int_c^x \frac{\lambda - q_0(t)}{q_1(t)} \, dt$$

for some constants $C = 0$, $c \neq u$. It is easily seen that since

$$\le \frac{q_1(x)}{(x-u)^n} \le A, \quad 1/A$$

for some constant A, f has a singularity at u and cannot be extended as an analytic function to the whole interval (a, b). Q

Proposition 3.5 *Let $q_1 : (a, b) \to \mathbb{R}$, $q_0 : (a, b) \to \mathbb{C}$, $-\infty \le a < b \le +\infty$, be real analytic and the operator D:*

$$D(f)(x) = q_1(x)f^1(x) + q_0(x)f(x)$$

be conjugate on some D invariant subspace of $A(a, b)$ to ϑ on some $A(k_-, k_+)$, $-\infty \le k_- < k_+ \le +\infty$, then q_1 has at most one zero $u \in (a, b)$ and $q_1^1(u) = 0$.

Proof: The result follows by Proposition 3.4 but we give another proof. Conjugation implies that D must have an eigenvector f_n for any eigenvalue n for every $n \in \mathbb{N}$. Thus for any $n \in \mathbb{N}$

$$q_1 f_n^1 = (n - q_0) f_n \text{ on } (a, b). \tag{12}$$

Let $q_1(u) = 0$ for some $u \in (a, b)$. Choose $n \in \mathbb{N}$ such that $q_0(u) \neq n$. Then (12) implies that $f_n(u) = 0$ and comparing the order of the zero u on both sides we see that $q_1^1(u) = 0$.

That q_1 has at most one zero on (a, b) follows from Remark 3.3. Q

Finally we characterize conjugation on invariant subspaces.

Theorem 3.6 *Let $q_1 : (a, b) \to \mathbb{R}$, $q_0 : (a, b) \to \mathbb{C}$, $-\infty \le a < b \le +\infty$, be real analytic. Then the general linear first order ordinary differential operator D,*

$$D(f)(x) := q_1(x)f^1(x) + q_0(x)f(x),$$

is conjugate on some D invariant subspace Y of $A(a, b)$ to ϑ on some $A(k_-, k_+)$, $-\infty \le k_- < k_+ \le +\infty$, if and only if either q_1 has no zeros or q_1 has exactly one zero $u \in (a, b)$ and $q_1^1(u) \neq 0$ moreover there are $0 < q \in \mathbb{N}$ and $p \in \mathbb{N}$, such that

$$q_1^1(u) = \frac{1}{q}, \qquad q_0(u) = -\frac{p}{q}.$$

If the first alternative holds then we can choose $0 \le k_- < k_+ \le +\infty$ and $Y = A(a, b)$. If the second alternative holds then $-\infty \le k_- < 0 < k_+ \le +\infty$ and $\operatorname{codim} Y = p$ if $q = 1$. In both cases conjugation can be given by a weighted composition operator, which is just a composition operator if and only if $q_0 \equiv 0$.

Proof: *Necessity.* By Proposition 3.5, q_1 has at most one zero which then is of order one.
Assume that $q_1(u) = 0$, $q_1'(u) = \lambda = 0$. By conjugation, D has an eigenvector f_n for the eigenvalue $n \in \mathbb{N}$ and for every $n \in \mathbb{N}$

$$q_1 f_n' = (n - q_0) f_n. \tag{13}$$

If $n \neq q_0(u)$ then $f_n(u) = 0$ by (13). Let $m(n)$ be the order of the zero of f_n at u. Comparing the coeffi cients of $(x-u)^{m(n)}$ in (13) as in (vi) in the proof of Theorem 3.2 we get

$$m(n)\lambda = n - q_0(u) \text{ for any } n \in \mathbb{N} \setminus \{q_0(u)\} \tag{14}$$

so

$$n - k = [m(n) - m(k)]\lambda \qquad \text{if } n, k \in \mathbb{N} \setminus \{q_0(u)\}. \tag{15}$$

Hence $0 \neq q := [m(n+1) - m(n)] \in \mathbb{Z}$ for large n. If $\lambda < 0$ then $m(n+1) < m(n)$ for large n, a contradiction, since $m(k) \in \mathbb{N}$ for any k. Hence $q \in \mathbb{N} \setminus \{0\}$ and $q_1'(u) = \lambda = 1/q$. This proves the claim if $q_0(u) = 0$.
Let $q_0(u) = 0$. Then we may apply (15) for $k = 0$ and large n and get

$$m(n) = m(0) + nq \text{ for large } n \tag{16}$$

and therefore (using also (16)) for large n

$$q_0(u) = n - m(n)\lambda = \frac{nq - m(0) - nq}{q} = -\frac{m(0)}{q} =: -p/q$$

proving the claim.
Sufficiency. In case q_1 has no zeros we apply Theorem 3.2.
Assume that $q_1(u) = 0$, $q_1'(u) = \frac{1}{q}$, $q_0(u) = -\frac{p}{q}$, $q_1(x) = (x-u)r(x)$, r never vanishing. Let

$$\kappa(x) := (x-u)^q \kappa_1^q(x) = (x-u)^q \exp\left(\int_u^x \frac{1 - qr(t)}{q_1(t)}\,dt\right),$$

$$w(x) := (x-u)^p w_1^p(x) = (x-u)^p \exp\left(\int_u^x \frac{-q_0(t) - pr(t)}{q_1(t)}\,dt\right).$$

Let us calculate κ':

$$\begin{aligned}
\kappa'(x) &= q(x-u)^{q-1}\kappa_1(x) + (x-u)^q \frac{1 - qr(x)}{q_1(x)}\kappa_1(x) \\
&= (x-u)^{q-1}\kappa_1(x)\left(q + \frac{1 - qr(x)}{r(x)}\right) \\
&= (x-u)^{q-1}\kappa_1(x)\frac{1}{r(x)}.
\end{aligned} \tag{17}$$

and hence

$$q_1\kappa' = \kappa \text{ and, similarly, } q_1 w' = -q_0 w.$$

This implies

$$\begin{aligned}
(D \circ C_{\kappa,w})(f) &= q_1[w(f \circ \kappa)]' + q_0 w(f \circ \kappa) = q_1 w'(f \circ \kappa) + q_1\kappa' w(f' \circ \kappa) + q_0 w(f \circ \kappa) \\
&= -q_0 w(f \circ \kappa) + w\kappa(f' \circ \kappa) + q_0 w(f \circ \kappa) = w\kappa(f' \circ \kappa) = (C_{\kappa,w} \circ \vartheta)(f).
\end{aligned}$$

Please note that κ is a diff eomorphism if and only if $q = 1$. If q is odd then κ is strictly increasing. Otherwise it is decreasing till u and then increasing. So for odd q we define $k_- := \lim_{x\to a^+}\kappa(x)$, $k_+ := \lim_{x\to b^-}\kappa(x)$ and since $\kappa(u) = 0$ thus $k_- < 0 < k_+$. For even q we define $k_+ := \max(\lim_{x\to a^+}\kappa(x), \lim_{x\to b^-}\kappa(x)) > 0$, while we take as k_- any number < 0. Clearly if $w \equiv 1$ then $p = 0$ and $q_0 \equiv 0$ (of course, if $q_0 \equiv 0$ then $p = 0$ as $q_0(u) = -\frac{p}{q}$).
If $q = 1$ then $Y = \operatorname{Im} V$ consists of those functions having a zero at u of order $\ge p$, hence $\operatorname{codim} Y = p$. Q

4 Generalized Euler partial differential operators

Applying the above results we can also consider the case of several variables. Here we take a families of real analytic functions $q_{1,1}, \ldots, q_{d,1}, q_{1,0}, \ldots, q_{d,0}, q_{j,1} : (a_j, b_j) \to \lambda_j\mathbb{R}, q_{j,0} : (a_j, b_j) \to \mathbb{C}$, analytic, $q_{j,1}$ with at most one zero u_j (and of order one), $q'_{j,1}(u_j) = \lambda_j \in \mathbb{C}, \lambda_j \neq 0$, $q_{j,0}(u_j) = 0$. For the function $q_{j,1}$ (if it has no zeros) or for $\frac{q_{j,1}}{\lambda_j}$ we can find corresponding analytic diffeomorphisms $\kappa_j, j = 1, \ldots, d$, of (a_j, b_j) onto (k_{j-}, k_{j+}) and weights w_j according to Theorem 3.2. Let us assume that for $j = 1, \ldots, m$ the function $q_{j,1}$ has no zeros and for $j = m + 1, \ldots, d$ it has a zero u_j and then $q_{j,0}(u_j) = 0$ for $j = m + 1, \ldots, d$. As usual we define

$$D_j(f)(x) = q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j)f(x).$$

Then for any entire function of exponential type zero E, $E(z) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha z^\alpha$, the following operator defined by

$$E(D_1, \ldots, D_d)(g)(x) = \sum_{\alpha \in \mathbb{N}^d} a_\alpha D_1^{\alpha_1} \cdots D_d^{\alpha_d}(g)(x)$$

on $A(\prod_{j=0}^{d}(a_j, b_j))$ is conjugate to $E(\vartheta_1, \ldots, \vartheta_m, \lambda_{m+1}\vartheta_{m+1}, \ldots, \lambda_d\vartheta_d)$ on $A(\prod_{j=0}^{d}(k_{j-}, k_{j+}))$ via the weighted composition operator $C_{\kappa,w}$, where

$$\kappa(x_1, \ldots, x_d) = (\kappa_1(x_1), \kappa_2(x_2), \ldots, \kappa_d(x_d)), \qquad w(x) = w_1(x_1) \cdots w_d(x_d).$$

We can also define generalized κ, w-multipliers, i.e., operators for which all functions of the form

$$\kappa_\alpha(x) := w(x)\kappa_1(x_1)^{\alpha_1} \cdots \kappa_d(x_d)^{\alpha_d}$$

are eigenvectors. It is easily seen that these operators are conjugate to classical multipliers — and thus their theory is "identical" with the theory of classical multipliers. So every theorem on surjectivity or on algebraic/topological structure (see the papers [5–10]) can be automatically transferred via conjugation from the classical multipliers to these generalized multipliers. Details are left to the reader but we give some examples using the papers [8] and [10].

It would be interesting to find a criterion for which given linear partial differential operator with analytic variable coefficients $P(\partial, x)$ the theory applies. The following is a simple necessary condition — unfortunately it is not fully effective since there is no general effective way to findeigenvalues and eigenvectors.

Proposition 4.1 *If a finite order linear partial differential operator with analytic coefficients*

$$P(\partial, x) = \sum_{\alpha \in \mathbb{N}^d, |\alpha| \le m} a_\alpha(x) \frac{\partial^\alpha}{\partial x^\alpha}$$

is a generalized Euler differential operator then there is a polynomial W such that for every $\alpha \in \mathbb{N}^d$, the value $W(\alpha)$ is an eigenvalue of $P(\partial, x)$ and there are corresponding eigenvectors f_α such that f_0 does not vanish at any point, for every $j = 1, \ldots, d$, f_{e_j}/f_0 is a real analytic diffeomorphism (here e_j means the j-th unit vector) and

$$q_{j,1} = \frac{(f_{e_j}/f_0)}{\frac{\partial}{\partial x_j}(f_{e_j}/f_0)}, \qquad q_{j,0} = -q_{j,1}\frac{\frac{\partial}{\partial x_j} f_0}{f_0}$$

satisfy the following conditions: $q_{j,1}$ is real valued, has at most one zero u_j and this zero is of order one.

Proof: By the very definition $P(\partial, x)$ is a generalized Euler partial differential operator if there exists first order linear ordinary differential operators with analytic coefficients $D_j, j = 1, \ldots, d$, conjugate with ϑ such that $P(\partial, x) = W(D_1, \ldots, D_d)$ for some polynomial W. Clearly $W(D_1, \ldots, D_d)$ is conjugate to $W(\vartheta_1, \ldots, \vartheta_d)$ and the latter has eigenvalues $W(\alpha)$ and by Theorem 3.2 with eigenvectors

$$f_\alpha(x_1, \ldots, x_d) = w(x)\kappa_1(x_1)^{\alpha_1} \cdots \kappa_d(x_d)^{\alpha_d},$$

where κ_j are real valued analytic diffeomorphisms and w is analytic complex valued without zeros. Now, $\kappa_j = f_{e_j}/f_0$ and $w = f_0$. The statements on $q_{j,1}$ and $q_{j,0}$ hold of Theorem 3.2 since

$$q_{j,1}(x_j) = \frac{\kappa_j(x_j)}{\kappa_j'(x_j)} \quad \text{and} \quad q_{j,0}(x_j) = -q_{j,1}(x_j)\frac{w_j'(x_j)}{w_j(x_j)}.$$

Indeed, the second equation holds since $D_j w = D_j f_0 = 0$. From this we get the first equation since $w\kappa_j = f_{e_j} = D_j f_{e_j} = D_j(w\kappa_j)$.

First, assume that $m = d$ so all $q_{1,1},\ldots, q_{d,1}$ have no zeros, then for every polynomial P of d variables by Theorem 3.2, $P(D_1,\ldots,D_d)$ on the space $A\left(\prod_{j=0}^{d}(a_j, b_j)\right)$ is conjugate to $P(\vartheta_1,\ldots,\vartheta_d)$ on $A\left(\prod_{j=0}^{d}(k_{j-}, k_{j+})\right)$ and $\Omega := \prod_{j=0}^{d}(k_{j-}, k_{j+}) \subset \mathbb{R}^d_+$. By [10, Section 10], the latter is conjugate to the partial differential operator with constant coefficients $P(\partial_1,\ldots,\partial_d)$ via $C_{\log}$, $C_{\log}(f)(x_1,\ldots,x_d) := f(\log x_1,\ldots,\log x_d)$, on $A\left(\prod_{j=0}^{d}(\log k_{j-}, \log k_{j+})\right)$. Since $\Omega_1 := \prod_{j=0}^{d}(\log k_{j-}, \log k_{j+})$ is convex we then can apply Hörmander's theory [12] characterizing the surjectivity of $P(\partial_1,\ldots,\partial_d)$ on $A(\Omega_1)$ by means of a Phragmen-Lindelöf - type condition valid on the characteristic variety of the principal part of P. For instance, by [10, Cor. 10.5], we get:

Corollary 4.2 *If $q_{j,1} : (a_j, b_j) \to \mathbb{R}$, $-\infty \le a_j < b_j \le +\infty$, $j = 1,\ldots,d$, are real analytic without zeros, $q_{j,0} : (a_j, b_j) \to \mathbb{C}$ are real analytic, $D_j(f)(x) = q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j)f(x)$, then for any polynomial P of d variables the operator*

$$P(D_1,\ldots,D_d) : A\left(\prod_{j=0}^{d}(a_j, b_j)\right) \to A\left(\prod_{j=0}^{d}(a_j, b_j)\right)$$

is surjective if and only if

$$P(\partial_1,\ldots,\partial_d) : A\left(\prod_{j=0}^{d}(\log k_{j-}, \log k_{j+})\right) \to A\left(\prod_{j=0}^{d}(\log k_{j-}, \log k_{j+})\right)$$

is surjective.

By [10, Cor. 10.6] we get:

Corollary 4.3 *Let P be a polynomial of d variables and of second order. Let $q_{j,1} : (a_j, b_j) \to \mathbb{R}$, $-\infty \le a_j < b_j \le +\infty$, $j = 1,\ldots,d$, be real analytic functions with $\int_{a_j}^{c_j}\frac{1}{q_j(x)}dx$ and $\int_{c}^{b_j}\frac{1}{q_j(x)}dx$ divergent for some $c_j \in (a_j, b_j)$, $j = 1,\ldots,d$, and let $q_{j,0}(a_j, b_j) \to \mathbb{C}$ be real analytic, $D_j(f)(x) = q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j)f(x)$. Then the operator $P(D_1,\ldots,D_d)$ is surjective on $A\left(\prod_{j=0}^{d}(a_j, b_j)\right)$ if and only if the principal part P_2 of the polynomial is either elliptic or proportional to a real indefinite quadratic form or to the product of two real linear forms.*

Now, consider the case when $q_{1,1},\ldots,q_{d,1}$ all have exactly one zero u_j of order one, $q_{j,1}'(u_j) =: \lambda_j \neq 0$. For simplicity we assume that $q_{j,1}$ is real valued and $\lambda_j > 0$ for any j. Moreover, $q_{j,0}(u_j) = 0$, $q_{j,0}$ are complex valued real analytic, $D_j(f)(x) = q_{j,1}(x_j)\frac{\partial f}{\partial x_j}(x) + q_{j,0}(x_j)f(x)$.

Let us recall that a polynomial P has the *halfplane property* if it does not vanish on any $z \in \mathbb{C}^d$ with $\mathrm{Re}\, z_1,\ldots, \mathrm{Re}\, z_d > 0$ (see [2] where properties of such polynomials and their characterizations are given, see also [8]). If the same holds for all $z \in \mathbb{C}^d$ with $\mathrm{Re}\, z_1,\ldots, \mathrm{Re}\, z_d \ge 0$ we say that P has the *closed halfplane property*. The latter property for homogeneous polynomials is equivalent to the halfplane property plus non-vanishing of P on the canonical unit vectors (see [8, Th. 8.5]).

There is an extensive literature on the halfplane property motivated partly by applications to image processing, see the survey paper [2]. In particular the following holds (for the authorship of the results below see the reference list in [2]).

Theorem 4.4 *(a) [2, Th. 9.1] Elementary symmetric polynomials have the halfplane property but its closed version if and only if the polynomial is of order one.*

(b) [2, Th. 8.1] If A is a complex $r \times d$ matrix then (a polynomial of order $\leq r$) $P(x) = \det(A \operatorname{diag} x A^*)$ has the halfplane property where $\operatorname{diag} x$ means the diagonal $d \times d$ matrix with coefficients $x_1, \dots, x_d$. It has its closed version only for $r = 1$.

(c) [2, Th. 5.3] A quadratic form has a halfplane property if and only if it is proportional to a form with all coefficients real non-negative and its matrix has exactly one strictly positive eigenvector. It has the closed halfplane property if and only if additionally all the diagonal coefficients of the latter matrix are strictly positive.

(d) [2, Cor. 2.9] Any partial derivative of a polynomial with the halfplane property has the same property.

(e) [2, Th. 6.1] If a homogeneous polynomial has the halfplane property then it is proportional to a polynomial with all coefficients real non-negative.

The paper [2] contains plenty of sufficient conditions and necessary conditions for the half-plane property (notice, for example, [2, Th. 7.2]) but the authors claim that there is still no effective algorithm to check this property.

Let us define for $I \subset \mathbb{N}^d$ and $u \in \Omega \subseteq \mathbb{R}^d$,

$$A_{I,u}(\Omega) := \{f \in A(\Omega) \mid f^{(\alpha)}(u) = 0 \text{ for } \alpha \notin I\} \quad \text{and} \quad A_I(\Omega) := A_{I,0}(\Omega).$$

Then we have the following observations.

Proposition 4.5 *Let P be a polynomial and P_m its principal part. Then for any $\lambda_1, \dots, \lambda_d > 0$ we have:*

(a) P has the halfplane property if and only if $P(\lambda_1 \cdot, \dots, \lambda_d \cdot)$ has the halfplane property.

(b) P_m has the closed halfplane property if and only if $P_m(\lambda_1 \cdot, \dots, \lambda_d \cdot)$ has the closed halfplane property.

(c) If $I \subseteq \mathbb{N}^d$ satisfies

$$\beta \notin I \text{ for any } \beta \leq \alpha,\ \alpha \notin I \tag{18}$$

then the weighted composition operator $C_{\kappa,w}$ maps A_I onto $A_{I,u}$ where

$$\kappa(x) = (\kappa_1(x_1), \dots, \kappa_d(x_d))$$

and κ_j analytic diffeomorphism with $\kappa_j(u_j) = 0$, w a complex weight without zeros.

Proof: It suffices to show only the part (c). We apply the Faá di Bruno formula:

$$\partial^\alpha(f \circ \kappa)(u) = \sum c_{\beta,\gamma} \partial^\beta f(\kappa(u)) \cdot \kappa^\gamma(u),$$

where $\beta \leq \alpha$ and the analogous formula holds also for κ^{-1}. This implies the result for $w \equiv 1$. Now, multiplication by w is an isomorphism of $A_{I,u}$ onto $A_{I,u}$. Q

Corollary 4.6 *Let $q_{j,1} : (a_j, b_j) \to \mathbb{R}$ be real analytic with exactly one zero $u_j \in (a_j, b_j)$ which is of order one and $q'_{j,1}(u_j) = \lambda_j > 0$, for $j = 1, \dots, d$. Let $q_{j,0} : (a_j, b_j) \to \mathbb{C}$ be real analytic with $q_{j,0}(u_j) = 0$, $D_j(f)(x) = q_{j,1}(x_j) \frac{\partial}{\partial x_j} f(x) + q_{j,0}(x_j) f(x)$. Let P be a polynomial of d variables, of order m. Let $\Omega := \prod_{j=0}^{d} (a_j, b_j)$ and $u = (u_1, \dots, u_d)$.*

(a) If the range of $P(D_1, \dots, D_d) : A(\Omega) \to A(\Omega)$ contains $A_{\beta + \mathbb{N}^d, u}(\Omega)$ then the following equivalent conditions hold:

(i) The range of $P_m(D_1, \dots, D_d) : A(\Omega) \to A(\Omega)$ contains $A_{1 + \mathbb{N}^d, u}(\Omega)$;

(ii) P_m has the halfplane property.

(b) *The following assertions are equivalent:*

(i) *The range of* $P_m(D_1,\ldots,D_d) : A(\Omega) \to A(\Omega)$ *contains* $A_{\mathbb{N}^d\setminus\{0\},u}(\Omega)$;

(ii) P_m *has the halfplane property and* P_m *does not vanish on the canonical unit vectors;*

(iii) *The range of* $P_m(D_1,\ldots,D_d) : A(\Omega) \to A(\Omega)$ *contains* $A_{I,u}(\Omega)$, *where* I *is cofinite and satisfies (18) and* P_m *does not vanish on the canonical unit vectors.*

(c) *If additionally* P_m *does not vanish on the canonical unit vectors, then the following assertions are equivalent:*

(i) $P(D_1,\ldots,D_d) : A(\Omega) \to A(\Omega)$ *is Fredholm;*

(ii) $P(D_1,\ldots,D_d) : A(\Omega) \to A(\Omega)$ *has a closed and finite codimensional range;*

(iii) P_m *has the (closed) halfplane property.*

Proof: In all cases $P(D_1,\ldots,D_d)$ is conjugate via $C_{\kappa,w}$ to $P(\lambda_1\vartheta_1,\ldots,\lambda_d\vartheta_d) : A(\Omega_1) \to A(\Omega_1)$, $\Omega_1 = {}^{d}_{j=0}(k_{j-}, k_{j+})$. Since $V(\Omega_1) \supseteq [0,1]^d$ we can apply [8, Th. 8.1] in (a), [8, Th. 8.5] in (b) and [8, Cor. 8.8] in (c). Please note that since "partial inverses" in the above results correspond to analytic functionals with support in $[0,1]^d$ these results hold for $A(\Omega_1)$, $V(\Omega_1) \supseteq [0,1]^d$ instead of $A(\mathbb{R}^d)$. The condition (18) in part (b) follows from the proof of [8, Th. 8.5 (e)⇒(f) ⇒(g)]. The conclusion follows by Proposition 4.5. Q

For polynomials of second order in two variables we get:

Corollary 4.7 *Same general assumption as in Corollary 4.6. Let* $P(z_1,z_2) := \sum_{|\alpha|\le 2} a_\alpha z_1^{\alpha_1} z_2^{\alpha_2}$ *be a polynomial of second order in two variables and assume that the principal part depends on the variable* z_1. *Then the following are equivalent:*

(i) *The range of* $P(D_1,D_2) : A(\Omega) \to A(\Omega)$ *contains* $A_{\beta+2\mathbb{N}^2,u}(\mathbb{R}^2)$ *for some* $\beta \in \mathbb{N}^2$;

(ii) $P(\vartheta)$ *is invertible on* $A_{I(P)}(\mathbb{R}^2)$ *with* $I(P) \supset \gamma + \mathbb{N}^2$ *for some* $\gamma \in \mathbb{N}^2$, *where* $I(P) = \{\alpha \mid P(\alpha) \,/\!= 0\}$;

(iii) *One of the following conditions hold:*

- P_2 *depends only on the variable* z_1 *and* $\mathrm{Re}\,\frac{a_{01}}{a_{20}} \ge 0$;
- P_2 *depends on both variables and* P_2 *has the halfplane property, i.e.,*

$$P_2(z_1,z_2) = C(b_{11}z_1 + b_{12}z_2)(b_{21}z_1 + b_{22}z_2), \quad \text{for } C \in \mathbb{C},\ b_{jk} \ge 0.$$

Proof: This follows as above using [8, Th. 10.7]. Q

A similar theory can be developed also on the basis of Theorem 3.6 giving surjectivity of the operator $P(D_1,\ldots,D_d)$ on a suitable invariant subspace of $A({}^{d}_{j=0}(a_j,b_j))$ or closed range of the operator $P(D_1,\ldots,D_d)$ containing a "big subspace" of the same suitable invariant subspace. The details are left to the reader.

To give some taste of the above results let us give a particular example.

Example 4.8 *The following differential operator is surjective on* $A(\mathbb{R}^3)$:

$$\begin{aligned} Z(f) &= (\arctan y)\frac{x}{x^2+1}\frac{\partial^2 f}{\partial x\partial y} + (e^z - 1)\arctan y\frac{\partial^2 f}{\partial y\partial z} + \frac{x}{x^2+1}(e^z-1)\frac{\partial^2 f}{\partial x\partial z} \\ &+ \frac{x}{x^2+1}(\cosh z + 1)\frac{\partial f}{\partial x} + (\arctan y)(\sin x + 2)\frac{\partial f}{\partial y} + 2(e^z-1)\frac{\partial f}{\partial z} \\ &+ 2(\cosh z - 1 + \sin x)f + 3f. \end{aligned}$$

Please note that $Z = P(D_1,D_2,D_3)$, where

$$P(x,y,z) = xy + yz + xz + 2(x+y+z) + 3$$

and

$$D_1(f)(x,y,z) := \frac{x}{x^2+1}\frac{\partial f}{\partial x}(x,y,z) + (\sin x)f(x,y,z),$$
$$D_2(f)(x,y,z) := \arctan y\,\frac{\partial f}{\partial y}(x,y,z),$$
$$D_3(f)(x,y,z) := (e^z-1)\frac{\partial f}{\partial z}(x,y,z) + (\cosh z-1)f(x,y,z).$$

By Theorem 3.2, $P(D_1, D_2, D_3)$ on $A(\mathbb{R}^3)$ is conjugate to $P(\vartheta_1, \vartheta_2, \vartheta_3)$ on $A(\mathbb{R}^3)$ and this is surjective by [8, Ex. 10.6] hence Z is also surjective.

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